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MULTIPLICITIES OF SUBGRAPHS

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A former conjecture of Burr and Rosta [1], extending a conjecture of Erdős [2], asserted that in any two-colouring of the edges of a large complete graph, the proportion of subgraphs isomorphic to a fixed graph G which are monochromatic is at least the proportion found in a random colouring. It is now known that the conjecture fails for some graphs G, including $G=K_p$ for $p\geq 4$.

We investigate for which graphs G the conjecture holds. Our main result is that the conjecture fails if G contains K_4 as a subgraph, and in particular it fails for almost all graphs.

1. Introduction

One of the most natural extremal graph theory questions which can be asked about a graph G is how many subgraphs isomorphic to G must be contained in every graph H of a specified order and size. This question has been studied in general for graphs and hypergraphs G by Erdős and Simonovits [4], though there is a substantial amount of earlier work.

The analogous and equally natural Ramsey theory question is how many monochromatic subgraphs isomorphic to G must be contained in any two-colouring of the edges of the complete graph K_n. It is this question that we shall be concerned with in this paper. Erdős [2] conjectured that for complete graphs G, random colourings will yield essentially the minimum number of monochromatic copies of G. The conjecture was proved by Goodman [7] for $G=K_3$, a better proof being given by Lorden [9]. Erdős' conjecture was extended to cover all graphs by Burr and Rosta [1]; we state this conjecture precisely in Section 3. The conjecture has been verified for certain graphs, in particular for cycles by Sidorenko [10], and we shall discuss this later. However, Sidorenko [10] also showed that the graph consisting of a triangle with a pendant edge fails to satisfy the Burr–Rosta conjecture, and Thomason [15] showed that $G=K_p$ fails to satisfy Erdős' conjecture for $p\geq 4$. We shall call graphs which satisfy the Burr–Rosta conjecture *common*, since they occur commonly in two-colourings.

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Given that the Burr–Rosta conjecture does not hold for all graphs it is of interest to determine which graphs are common. The main result of this paper is that any graph containing K_4 is uncommon, and in particular that almost all graphs are uncommon. As well as proving this result we shall discuss whether other graphs are common. In this matter the paper takes on more of the nature of a brief survey, though we make a few new contributions.

It is conjectured that the Burr–Rosta conjecture holds for all bipartite graphs, and that in this case the problem is not one of Ramsey theory at all but one of extremal graph theory, as described in the opening paragraph. In other words, it is conjectured that if G is a bipartite graph then every large graph H contains at least as many copies of G as does a random graph of the same order and density as H. This conjecture would imply that every bipartite graph is common. This fascinating and beautiful conjecture, which is the analogue in extremal graph theory of the Burr–Rosta conjecture, has been made in its strongest form by Sidorenko [11], who has also made the most progress towards establishing it. We will discuss this further in Section 2.

In the light of the bipartite graph conjecture and our result about graphs containing K_4 it is natural to ask whether there is a relation between the truth of the Burr–Rosta conjecture and the clique number or chromatic number of G. However, as we shall show later, there are uncommon triangle-free graphs, so there is no apparent relationship between commonality and the clique number. Nevertheless, it seems entirely possible that all 2-chromatic graphs are common and all graphs with chromatic number at least 4 are uncommon. Examples of both common and uncommon graphs of chromatic number 3 are known. The class of wheels form an interesting case. We will show that wheels with an even number of spokes, which are 3-chromatic graphs, are common. The odd wheels are 4-chromatic graphs, and apart from the 3-wheel (which is just K_4), we do not know the commonality of any odd wheel.

2. Bipartite Graphs

Let G and H be graphs. A *homomorphic copy* of G in H is a map $f: V(G) \to V(H)$ such that $f(u)f(v) \in E(H)$ whenever $uv \in E(G)$. We denote by $\mathrm{Hom}(G;H)$ the number of homomorphic copies of G in H. The following conjecture appears in Sidorenko [12] and in a much weaker form in Erdős and Simonovits [4]. Essentially it states that the minimum number of homomorphic copies of G is attained by a random graph H.

Conjecture 1. *Let G be a bipartite graph and let H be a graph of order n and average degree pn, $0 \le p \le 1$. Then $Hom(G;H) \ge n^{|G|}p^{e(G)}$.*

Since the number of homomorphic copies of G which are not injective is of order $O(n^{|G|-1})$, the conjecture would imply that the number of labelled subgraphs of H isomorphic to a fixed labelled bipartite graph G is at least $n^{|G|}p^{e(G)} + O(n^{|G|-1})$.

The advantage of considering homomorphic images of G is that the conjectured inequality is devoid of error terms and applies for all values of n, p and $|G|$.

Stated differently, the conjecture says that the probability $p(G)$ of a randomly chosen map $f : V(G) \to V(H)$ being a homomorphic copy of G is at least $p^{e(G)}$. Sidorenko [13] has constructed a strengthened version of the conjecture, involving arbitrary weight functions on the vertex set.

The conjecture is true for bipartite graphs G which are stars (trivially), complete bipartite graphs (essentially due to Erdős and Moon [3]), trees (Sidorenko [11]) and cycles of even length (Sidorenko [12]). An earlier proof given for trees by Sidorenko [14] appears to be slightly flawed in its published form. Here we give a slightly different proof.

Theorem 1. *Let T be a tree and let H be a graph of order n and average degree pn. Then $Hom(T;H) \geq n^{|T|}p^{|T|-1}$.*

Proof. Fix a vertex x of T, and let d be the degree of x. Let $xy_1,\dots,xy_d$ be the edges incident with x, and let $T_1,\dots,T_d$ be the subtrees of T such that T_i contains y_i and T consists of $T_1,\dots,T_d$ along with $xy_1,\dots,xy_d$. For each vertex $v \in V(H)$ let $t(v)$ be the number of homomorphic copies f of T in H such that $f(x)=v$, and let $t_i(v)$ be the number of of homomorphic copies f_i of T_i in H such that $f_i(y_i)=v$. Now let T^* be the tree obtained from the subtrees $T_1,\dots,T_d$ by identifying the vertices $y_1,\dots,y_d$ with a new vertex y, and adding d extra pendant edges to y so that $|T^*|=|T|$. Denote by $t^*(v)$ the number of homomorphic copies f^* of T^* in H such that $f^*(y)=v$. For $u,v \in V(H)$ let $a(u,v)=1$ if $u,v \in E(H)$ and let $a(u,v)=0$ otherwise. Then

$$\begin{aligned}
\sum_{v\in V(H)} t(v)^{1/d} &= \sum_{v\in V(H)} \left\{ \prod_{i=1}^{d} \left(\sum_{u\in V(H)} t_i(u)a(u,v) \right) \right\}^{1/d} \\
&\geq \sum_{v\in V(H)} \sum_{u\in V(H)} \prod_{i=1}^{d} t_i(u)^{1/d} a(u,v)^{1/d} \quad \text{by Hölder's Inequality} \\
&= \sum_{u\in V(H)} \prod_{i=1}^{d} t_i(u)^{1/d} d_H(u) = \sum_{u\in V(H)} t^*(u)^{1/d},
\end{aligned}$$

where $d_H(u)$ is the degree of u in H.

Let d^* be the degree of y in T^*. Then, $d^* \geq d$, so by Jensen's Inequality,

$$\left(\frac{1}{n} \sum_{v\in V(H)} t(v)^{1/d} \right)^{d} \geq \left(\frac{1}{n} \sum_{u\in V(H)} t^*(u)^{1/d} \right)^{d} \geq \left(\frac{1}{n} \sum_{u\in V(H)} t^*(u)^{1/d^*} \right)^{d^*}.$$

By repeating the process which converts T to T^* we can transform T to a star S with a root vertex z of degree $|T|-1$. Let $s(v)$ be the number of homomorphic copies of S rooted at $v \in V(H)$; thus $s(v) = d_H(v)^{|T|-1}$. Therefore

$$\begin{aligned}
\mathrm{Hom}(T;H) = \sum_{v\in V(H)} t(v) \quad &\geq \quad n\left(\frac{1}{n}\sum_{v\in V(H)} t(v)^{1/d}\right)^{d} \\
&\geq n\left(\frac{1}{n}\sum_{v\in V(H)} t^*(v)^{1/d^*}\right)^{d^*} \geq \cdots \geq n\left(\frac{1}{n}\sum_{v\in V(H)} s(v)^{1/(|T|-1)}\right)^{|T|-1} \\
&= n\left(\frac{1}{n}\sum_{v\in V(H)} d_H(v)\right)^{|T|-1} \geq n(pn)^{|T|-1},
\end{aligned}$$

as claimed. ∎

Sidorenko [13] showed that Conjecture 1 is true for bipartite graphs with at most four vertices in one of its vertex classes. He also showed the following. Let G_1 and G_2 be two bipartite graphs. Let F be an independent set in G_1 and let $a \in V(G_1)\backslash F$. Suppose that one of the vertex classes of G_2 has at most k elements. Form the graph G by taking k copies of G_1, identified in the vertices of F, and identifying one of the vertex classes of G_2 with some of the copies of the vertex a. Sidorenko [13] showed that if G_1 and G_2 satisfy his stronger conjecture, so does G. This implies in particular that Conjecture 1 is satisfied by trees, even cycles and complete bipartite graphs.

Here we demonstrate a variant of this result in which F need not be an independent set but may span at most one edge. (We thank the referee for pointing out that the same proof would work when F contains only independent edges.)

Theorem 2. *Let G be a graph satisfying Conjecture 1. Let F be a subgraph of G containing at most one edge. Let G^* be the graph obtained from k disjoint copies of G by identifying the k copies of F. Then G^* satisfies Conjecture 1.*

Proof. Let H be a graph of order n and average degree pn. Let $\mathcal{F}$ be the set of homomorphic copies of F in H. If F contains exactly one edge then $|\mathcal{F}| = n.pn.n^{|F|-2} = pn^{|F|}$, and otherwise $|\mathcal{F}| = n^{|F|}$; in both cases $|\mathcal{F}| = n^{|F|}p^{e(F)}$. For each $f \in \mathcal{F}$, let $d(f)$ be the number of homomorphic copies of G in H whose restriction to $V(F)$ is f. Let d be the average value of $d(f)$ over $f \in \mathcal{F}$; then

$$d|\mathcal{F}| = \sum_{f\in\mathcal{F}} d(f) = \mathrm{Hom}(G;H) \geq n^{|G|}p^{e(G)}.$$

It follows that

$$\mathrm{Hom}(G^*;H) = \sum_{f\in\mathcal{F}} d(f)^k \geq |\mathcal{F}|d^k$$

$$\geq |\mathcal{F}|^{1-k} n^{k|G|} p^{ke(G)}$$
$$= n^{k|G|-k|F|+|F|} p^{ke(G)-ke(F)+e(F)}$$
$$= n^{|G^*|} p^{e(G^*)},$$

as claimed. ∎

We can extend the discussion in this section to cover r-uniform r-partite hypergraphs. If G and H are r-uniform hypergraphs, a homomorphic copy of G in H is defined to be a map $f : V(G) \to V(H)$ such that $f(e) \in E(H)$ whenever $e \in E(G)$. Sidorenko [12] showed that the natural extension of Conjecture 1 to all r-partite r-uniform hypergraphs is false for $r \geq 3$. However he does prove it for r-trees. The class of r-trees is defined recursively; a single (hyper) edge is an r-tree, and a graph obtained from an r-tree by adding a new edge and identifying some of the new vertices with a subset of an existing edge is also an r-tree.

Theorem 3. (Sidorenko [11]) *Let H be an r-uniform hypergraph with $pn^r/r!$ edges and let G be an r-tree. Then $Hom(G;H) \geq n^{|G|} p^{e(G)}$.*

We shall make use of this theorem in the next section.

3. Ramsey Multiplicities

Given a graph H of order n we may associate with H a colouring of the complete graph K_n on the vertex set $V(H)$, wherein the edges of K_n which are in H are coloured red and those which are in the complementary graph $\overline{H}$ are coloured blue. Now let G be a graph. A *monochromatic copy of G in H* is an injective map $g: V(G) \to V(H)$ such that either $g(u)g(v) \in E(H)$ whenever $uv \in E(G)$ (these are the red copies of G), or $g(u)g(v) \in E(\overline{H})$ whenever $uv \in E(G)$ (these are the blue copies of G). Note that the red copies of G in H are homomorphic copies of G in H in the sense of the previous section, though the converse is false since here we are requiring that g be injective. Note also that we are in effect counting *labelled* copies of G in H. For example, if $G = K_3$ then each set of three vertices forming a triangle in H (which we call an *unlabelled* triangle) will give rise to six red copies of G in H.

We denote by $R(G;H)$ the set of red copies of G in H, and by $B(G;H)$ the set of blue copies; of course, $B(G;H) = R(G;\overline{H})$. If G has at least one edge then $R(G;H) \cap B(G;H) = \emptyset$ but if $E(G) = \emptyset$ then $R(G;H) = B(G;H)$. Note that $|R(G;H)| = \mathrm{Hom}(G;H)(1+o(1))$, the $o(1)$ term indicating a term which is small when $|H|$ is large. We further denote by $c(G;H)$ the proportion of labelled subgraphs of H isomorphic to G which are monochromatic; that is,

$$c(G;H) = \left(|R(G;H)| + |B(G;H)|\right) / n(n-1)\ldots(n-|G|+1).$$

We then define $c(G;n)$ to be the minimum of $c(G;H)$ over all graphs H of order n. It is not difficult to show that $c(G;n)$ is an increasing function of n which is bounded above by two. Therefore $c(G;n)$ tends to a limit as $n\to\infty$, and we denote this limit by $c(G)$.

The average value of $c(G;H)$ over all graphs H is $2^{1-e(G)}$. It follows that $c(G)\le 2^{1-e(G)}$. We shall call a graph *common* if $c(G)=2^{1-e(G)}$. If $c(G)<2^{1-e(G)}$ then there must be colourings in which monochromatic copies of G are less frequent than on average, and we call such graphs G *uncommon*. The question of whether a graph G is common or uncommon we shall call the question of the *commonality* of G.

It was conjectured by Erdős [2] that complete graphs K_p are common; the truth of this conjecture for $p\le 3$ follows from Goodman's theorem [7]. Burr and Rosta [1] extended the conjecture to cover all graphs. However, Sidorenko [10] disproved Burr and Rosta's conjecture by showing that a triangle with a pendant edge is uncommon. Moreover, Thomason [15] showed that K_p is uncommon for $p\ge 4$, so even the conjecture of Erdős fails to hold.

It is therefore interesting to investigate which graphs are common and which are uncommon. The first thing to notice is that the truth of Conjecture 1 would imply that *every bipartite graph is common.* For, let G be a graph satisfying the conjecture, and let (H_n) be a sequence of graphs satisfying $c(G;H_n) = c(G;n)$. If H_n has order n and average degree $p_n n$, then $\mathrm{Hom}(G;H_n) \ge n^{|G|}p_n^{e(G)}$ and $\mathrm{Hom}(G;\overline{H}_n)\ge n^{|G|}q_n^{e(G)}$, where $\overline{H}$ is the complement of H and $p_n n+q_n n=n-1$. Therefore

$$c(G;n) \ge (p_n^{e(G)} + q_n^{e(G)})(1+o(1)) \ge 2^{1-e(G)}(1+o(1)),$$

so G is common. Consequently we know that trees, even cycles and complete bipartite graphs are common. We are aware of no bipartite graphs which are known to be common for which Conjecture 1 has not also been verified.

From now on we consider only non-bipartite graphs. The simplest such graphs, namely the odd cycles, were shown to be common by Sidorenko [10] using an elegant argument. The next simplest cases consist of odd cycles plus one extra edge. The edge can be added either as a chord or as a pendant edge. We are at present unable to resolve the status of odd cycles with chords; even the commonality of a pentagon with a chord is unknown. However, odd cycles with pendant edges are known to be uncommon. In fact, any odd cycle to which pendant trees have been attached is uncommon. Moreover, a similar phenomenon pertains to any non-bipartite graph; the result of adding pendant trees produces an uncommon graph, at least provided that the trees are sufficiently large. (By the phrase 'the graph G is formed by adding a pendant tree T to the graph F', we mean that T is a tree with $|V(T)\cap V(F)|=1$ and G is the union of F and T.)

Theorem 4. *Let $m\ge 3$. Then there exists a number $t_0=t_0(m)$ such that if F is any non-bipartite graph of size m, and if G is any graph obtained from F by adding pendant trees whose sizes total at least t_0, then G is uncommon.*

Proof. We shall show that $t_0(m) = \lceil 4^m m \log 2 \rceil$ will work. Let G be formed from F by adding pendant trees of total size $t \geq t_0$. Consider a sequence (H_n) of graphs of order n with n even, where H_n is the disjoint union of two random graphs H_n^1 and H_n^2 each of order $n/2$. The edges in H_n^1 and H_n^2 will be chosen independently at random with probability $p = 1 - q$, where $q = (m/t)\log 2$. Note that

$$(1) \qquad (1-q)^{m+t} < (1-q)^t < e^{-qt} = 2^{-m}.$$

Moreover, $q \leq (m/t_0)\log 2 \leq 4^{-m}$, so

$$(2) \qquad (1+q)^t q < e^{qt} q < 2^m 4^{-m} = 2^{-m}.$$

Let us estimate the expectation $\mathbf{E}(|R(G;H_n)| + |B(G;H_n)|)$. Clearly

$$\mathbf{E}(|R(G;H_n)|) \leq 2(n/2)^{|G|}(1-q)^{e(G)} = 2(n/2)^{|F|+t}(1-q)^{m+t}.$$

To estimate $|B(G;H_n)|$, observe that there are $2^{|F|}$ ways to partition the vertices of F into two parts. Since F is not bipartite, for every one of these partitions at least one of the two parts spans an edge. Therefore, for a given partition, the expected number of copies of F in H, in which the vertices of one part lie in $V(H_n^1)$ and the vertices of the other lie in $V(H_n^2)$, is at most $(n/2)^{|F|}q$. Summing over all partitions of $V(F)$, we obtain

$$\mathbf{E}(|B(G;H_n)|) \leq 2^{|F|}(n/2)^{|F|}q\,[(1+q)(n/2)]^t = 2^{|F|}(n/2)^{|F|+t}(1+q)^t q.$$

Now, by (1) and (2), we obtain

$$\mathbf{E}(|R(G;H_n)| + |B(G;H_n)|) \leq n^{|F|+t}\left[2^{1-|F|-t-m} + 2^{-t-m}\right] \leq \frac{3}{4} n^{|F|+t} 2^{1-t-m}.$$

Therefore there is a sequence of graphs (H_n) with $c(G;H_n) \leq (3/4)2^{1-e(G)}$. Thus $c(G) < 2^{1-e(G)}$ and so G is uncommon. ∎

Remark. Clearly the estimated bound on the number of edges in the trees added to F to form G is very crude, and can be reduced greatly for any given graph F whose structure is known. In particular, if F is an odd cycle a slightly more careful argument shows that G need only have one more edge than F.

Remark. Note that if t and q are suitably chosen in the proof of Theorem 4 we can obtain graphs G for which the value of $c(G)/2^{1-e(G)}$ is arbitrarily small.

So far we have only been considering the commonality of connected graphs, and it is not really worth considering unconnected graphs. Whilst it is clear that there are common graphs whose vertex disjoint union is common, it is also possible to find two common graphs whose vertex disjoint union is uncommon. For example, the graphs in Theorem 4 show that the graph consisting of a triangle and an edge is in fact uncommon. If we take the union of a common graph with an uncommon one, the result may be uncommon (the union of a tree and a triangle with a pendant edge is shown uncommon by the colourings used in the proof of Theorem 4), but at present we cannot decide whether the result is always uncommon. Nor can we decide whether the disjoint union of two uncommon graphs is always uncommon. If we merely consider edge disjoint unions of graphs then the result may be common or uncommon.

The following theorem is analogous to Theorem 2 and its proof is very similar.

Theorem 5. *Let G be a common graph. Let F be a subgraph of G containing at most one edge. Let G^* be the graph obtained from k disjoint copies of G by identifying the k copies of F. Then G^* is common.*

Proof. Let (H_n) be a sequence of graphs with $|H_n| = n$. Let $\mathcal{F}_n = R(F; H_n) \cup B(F; H_n)$ be the set of monochromatic copies of F in H_n. For each $f \in \mathcal{F}_n$ let $r(f)$ be the number of red copies $g \in R(G; H_n)$ of G in H_n whose restriction to $V(F)$ is f. Let $b(f)$ be defined similarly by the blue copies of G. Note that if $e(F)=1$ then either $r(f)=0$ or $b(f)=0$, since the colour of any copy of G will be determined by the colour of the edge of F. Let d_n denote the average value of $r(f)+b(f)$ over $f \in \mathcal{F}_n$. Then

$$d_n|\mathcal{F}_n| = \sum_{f\in\mathcal{F}_n} r(f)+b(f) = c(G; H_n)n(n-1)\ldots(n-|G|+1) \geq 2^{1-e(G)}n^{|G|}(1+o(1)).$$

Observe that $r(f)^k+b(f)^k=(r(f)+b(f))^k$ if $e(F)=1$, and $r(f)^k+b(f)^k \geq 2((r(f)+b(f))/2)^k$ if $e(F)=0$. It follows that

$$\begin{aligned} c(G^*; H_n)n(n-1)\ldots(n-|G^*|-1) &= \sum_{f\in\mathcal{F}_n} (r(f)^k + b(f)^k)(1+o(1)) \\ &\geq \sum_{f\in\mathcal{F}_n} 2^{1-e(F)} \left(\frac{r(f)+b(f)}{2^{1-e(F)}}\right)^k (1+o(1)) \\ &\geq 2^{(e(F)-1)(k-1)}|\mathcal{F}_n|d_n^k(1+o(1)) \\ &\geq 2^{(e(F)-1)(k-1)}|\mathcal{F}_n|^{1-k}2^{k-ke(G)}n^{k|G|}(1+o(1)) \\ &= 2^{1-e(G^*)}n^{|G^*|}(1+o(1)). \end{aligned}$$

Therefore $c(G^*) \geq 2^{1-e(G^*)}$, and so G^* is common. ∎

The reason we can prove Theorem 5 (and for that matter Theorem 2) easily is that when $e(F) \leq 1$ the number of copies of F is independent of the structure of the underlying graph H. Clearly if $e(F) \geq 2$ the number of copies of F will depend on the colouring. Nevertheless we can still hope to obtain a result similar to Theorem 5 if we have a lower bound on the number of monochromatic copies of G in terms of the number of copies of F. In particular, Lorden's proof of Goodman's theorem gives such a bound when G is a triangle and F is a path of length two. Theorem 3 enables us to extend this to certain cases wherein G is a tree made up of triangles.

A *triangular-vertex-tree* is defined recursively; a single triangle is a triangular-vertex-tree, and any graph obtained by identifying a single vertex of a new triangle with a vertex of a triangular-vertex-tree will itself be a triangular-vertex-tree. Similarly, a single triangle is a *triangular-edge-tree*, and any graph obtained by identifying a single edge of a new triangle with an edge of a triangular-edge-tree will itself be a triangular-edge-tree.

The following argument is due to Sidorenko [11].

Theorem 6. (Sidorenko [11]) *Let H be a graph of order n and let Δ be the number of labelled monochromatic triangles in the colouring associated with H. Let Tv be a triangular-vertex-tree and Te be a triangular-edge-tree. Then*

$$c(Tv;H) \geq 2\left(\frac{\Delta}{2n^3}\right)^{e(Tv)/3}(1+o(1)) \text{ and } c(Te;H) \geq \left(\frac{\Delta}{n^3}\right)^{(e(Te)-1)/2}(1+o(1)),$$

where $o(1)$ indicates a term which is small when n is large.

Proof. Let $r=|R(K_3;H)|/n(n-1)(n-2)$ and $b=|B(K_3;H)|/n(n-1)(n-2)$; thus $r+b=\Delta n^{-3}(1+o(1))$. Let H_1 be the 3-uniform hypergraph of order n whose edges correspond to the red triangles, let H_2 be the analogous hypergraph formed by the blue triangles, and let $H_3=H_1\cup H_2$. To each of Tv and Te we may associate the obvious 3-trees $Tv3$ and $Te3$. By applying Theorem 3 to each of H_1, H_2 and H_3 we obtain

$$\begin{aligned} c(Tv;H) &= [\mathrm{Hom}(Tv3;H_1)+\mathrm{Hom}(Tv3;H_2)]\, n^{-|Tv|}(1+o(1)) \\ &\geq \left[r^{e(Tv3)}+b^{e(Tv3)}\right] n^{|Tv3|}n^{-|Tv|}(1+o(1)) \\ &\geq 2\left((r+b)/2\right)^{e(Tv3)}(1+o(1)) = 2\left(\Delta/2n^3\right)^{e(Tv)/3}(1+o(1)). \end{aligned}$$

Since no edge of H_1 has two vertices in common with an edge of H_2, it follows that any copy of $Te3$ in H_3 corresponds to a copy either in H_1 or in H_2. Therefore

$$\begin{aligned} c(Te;H) &= \mathrm{Hom}(Te3;H_3)n^{-|Te|}(1+o(1)) \\ &\geq \left(\Delta n^{-3}\right)^{e(Te3)} n^{|Te3|}n^{-|Te|}(1+o(1)) \\ &= \left(\Delta n^{-3}\right)^{(e(Te)-1)/2}(1+o(1)), \end{aligned}$$

as claimed. ∎

Goodman's theorem [7] implies that triangles are common, so in Theorem 6 the lower bound $\Delta n^{-3} \geq 1/4$ can be substituted. Consequently all triangular-vertex-trees and triangular-edge-trees are common. We are unable to say in general whether other tree-like structures formed from triangles are common. Even the commonality of the graph of order seven, formed from three triangles by joining two along an edge with the third sharing a vertex with one of the other two, is undecided. Sidorenko [11] proves in a similar way a more general version of Theorem 6, that tree-like structures formed from k-cycles joined at edges are common.

To apply Theorem 6 in other situations we need the following observation, used by Lorden [9] in his proof of Goodman's theorem.

Lemma 7. *Let Δ denote the number of labelled monochromatic triangles and $\mathcal{P}_n$ the set of labelled monochromatic paths of length two in some two-colouring of the edges of the complete graph of order n. Then*

$$2\Delta = 3|\mathcal{P}_n| - n(n-1)(n-2).$$

Proof. Each unlabelled path of length two lies in a unique unlabelled triangle. The $\Delta/6$ monochromatic triangles each contain three unlabelled monochromatic paths, the $\binom{n}{3} - \Delta/6$ others contain one monochromatic path. Therefore $|\mathcal{P}_n|/2 = 3\Delta/6 + \binom{n}{3} - \Delta/6$. ∎

The *k-wheel* W_k is the graph formed from a cycle of length k by adding a new vertex (the *hub*) and joining it to all other vertices. We can regard an even wheel W_{2k} as the union of two *k-fans* sharing a path of length two; this path consists of the hub and edges joining it to two diametrically opposite vertices of the $2k$-cycle. The two k-fans are triangle-edge-trees each formed from k triangles and are said to *rest* on the common path. Note that W_{2k} has $2k+1$ vertices and $4k$ edges, whilst the k-fan has $k+2$ vertices and $2k+1$ edges.

Theorem 8. *The even wheel W_{2k} is common for $k \geq 2$.*

Proof. We mentioned that a wheel can be regarded as the union of two k-fans F_k whose intersection is a path of length two. Let (H_n) be a sequence of graphs such that $|H_n| = n$ and $c(W; H_n) = c(W; n)$, where $W = W_{2k}$. Let P_2 be a path of length two and let $\mathcal{P}_n = R(P_2; H_n) \cup B(P_2; H_n)$. Moreover let $p_n = |\mathcal{P}_n| n^{-3}$. Denoting by $(d_i)_{i=1}^n$ the degree sequence of H_n, we observe that

$$|\mathcal{P}_n| = \sum_{i=1}^{n} \left[d_i^2 + (n - d_i)^2 \right] (1 + o(1)) \geq n^3/2(1 + o(1)),$$

so that $1/2 \leq p_n(1 + o(1)) \leq 1$. Let Δ_n denote the number of monochromatic triangles; by Lemma 7 we have

$$2\Delta_n n^{-3}(1 + o(1)) = (3|\mathcal{P}_n| n^{-3} - 1)(1 + o(1)) = (3p_n - 1)(1 + o(1)).$$

For each $P \in \mathcal{P}_n$ let $d(P)$ be the number of labelled monochromatic k-fans resting on P, and let d denote the average value of $d(P)$ over $P \in \mathcal{P}_n$. Therefore $d = |\Phi_n|/|\mathcal{P}_n|$, where Φ_n denotes the set of monochromatic k-fans. Then by Theorem 6,

$$d = |\Phi_n|/|\mathcal{P}_n| \geq (\Delta_n n^{-3})^k n^{k+2} |\mathcal{P}_n|^{-1} (1+o(1)) = p_n^{-1}(3p_n - 1)^k 2^{-k} n^{k-1}(1+o(1)).$$

Therefore we have, by Lemma 7, that

$$\begin{aligned} c(W;n)n(n-1)\ldots(n-2k) &= \sum_{P\in\mathcal{P}_n} d(P)^2(1+o(1)) \\ &\geq |\mathcal{P}_n|d^2(1+o(1)) \\ &\geq p_n^{-1}(3p_n-1)^{2k}2^{-2k}n^{2k+1}(1+o(1)). \end{aligned}$$

We may now conclude that $c(W)$ is bounded below by the minimum value of the function $2^{-2k}p^{-1}(3p-1)^{2k}$ for p in the range $1/2 \leq p \leq 1$. It is easily checked that this minimum is attained when $p=1/2$, giving the bound $c(W) \geq 2^{1-4k}$, as required. ∎

Arguments analogous to those in the proof of Theorem 8 can be used to demonstrate that various other graphs are common (for details see Jagger [8]).

Note that the odd wheels, W_{2k+1}, which are not covered by Theorem 8, have chromatic number four. We know of no graphs with chromatic number more than three which are common. Indeed, $W_3 = K_4$ is uncommon, and we shall show that every graph containing K_4 is uncommon. We regard the determination of the commonality of W_5 as the most interesting open problem in the area.

4. Graphs containing K_4

As remarked earlier, the graph K_4 is uncommon. Colourings demonstrating this fact were found by Thomason [15]. They consist of taking a certain fixed graph J and letting H_n be the m-fold cover of J whenever n is a multiple of $|J|$. The *m-fold cover* $m\circ J$ of J, where $V(J)=\{v_1,\ldots,v_{|J|}\}$, is the graph consisting of $m|J|$ vertices partitioned into $|J|$ classes $V_1,\ldots,V_{|J|}$ each of size m. The edges of $m\circ J$ are all those pairs xy such that $x\in V_i$, $y\in V_j$ and $v_iv_j\in E(J)$, $1\leq i<j\leq |J|$.

It is easy to calculate the properties which J needs to have in order that the colourings $m\circ J$ will show that K_4 is uncommon. The graphs used in [15], denoted T_k^-, were constructed from finite geometries and will be described below; these are the graphs we will use to show that any graph containing a K_4 is uncommon. Once it is realised that m-fold covers yield good colourings it is possible to use simpler graphs than T_k^-. One simple example is mentioned in [15]. A still simpler example is described by Franek and Rödl [5]. However, to show that K_p is uncommon for $p\geq 5$ it is necessary to have a sequence of suitable graphs J whose orders tend to infinity, and the only known sequence for this case consists of the graphs T_k^-.

The analysis of the graphs T_k^- in [15] is rather opaque and reflects the process by which the associated colourings were discovered. It is possible to extend that analysis to investigate the number of monochromatic copies of non-complete graphs G, but the calculations become almost impenetrable.

This situation was transformed when the second author of this paper made an entirely new analysis. He considered the graphs $m\circ T_k^-$ directly rather than working with T_k^- itself, and showed that the required calculation is essentially one of Fourier analysis. In this manner it is possible to give a far more elegant and transparent proof that K_p is uncommon for $p\geq 4$, and the proof easily extends to all graphs G which contain K_4. We shall now give that proof.

Let J be a graph with vertex set $\{v_1,\ldots,v_{|J|}\}$, and let $\psi : V(J)\times V(J)\to\{-1,1\}$ be defined by $\psi(v_i,v_j)=-1$ if $v_iv_j\in E(J)$ and $\psi(v_i,v_j)=1$ otherwise (note that $\psi(v_i,v_i)=1$ for all v_i). Let H be the m-fold cover $m\circ J$. We define a corresponding function $\tilde{\psi} : V(H)\times V(H)\to\{-1,1\}$ by $\tilde{\psi}(x,y)=\psi(v_i,v_j)$ if $x\in V_i$ and $y\in V_j$. Note that $\tilde{\psi}(x,y)=-1$ if $xy\in E(H)$ and $\tilde{\psi}(x,y)=1$ otherwise.

Let G be a fixed graph with vertex set $\{1,2,\ldots,p\}$. Consider the expression

$$\sum_{x_1\in V(H)}\cdots\sum_{x_p\in V(H)}\left[\prod_{ij\in E(G)}(1+\tilde{\psi}(x_i,x_j))+\prod_{ij\in E(G)}(1-\tilde{\psi}(x_i,x_j))\right]. \tag{3}$$

When $x_1,\ldots,x_p$ are distinct vertices of H, the first product will be zero unless these vertices span a subgraph isomorphic to G in the complementary graph $\overline{H}$, and the second product will likewise be zero unless the vertices span a copy of G in H. Products which are not zero are equal to $2^{e(G)}$. It follows that the whole expression (3) is equal to

$$2^{e(G)}c(G;H)n(n-1)\ldots(n-p+1)+O(n^{p-1}), \tag{4}$$

where $n=|H|=m|J|$.

Given that H is the m-fold cover of J it is clear that expression (3) is equal to

$$m^p\sum_{x_1\in V(J)}\cdots\sum_{x_p\in V(J)}\left[\prod_{ij\in E(G)}(1+\psi(x_i,x_j))+\prod_{ij\in E(G)}(1-\psi(x_i,x_j))\right]. \tag{5}$$

If we expand one of the products in expression (5) we get $2^{e(G)}$ terms, one for each spanning subgraph F of G. The term corresponding to F will be either doubled or cancelled by the corresponding term from the other product, according to whether $e(F)$ is even or odd. We therefore define

$$\Psi(J;F)=\frac{1}{|J|^p}\sum_{x_1\in V(J)}\cdots\sum_{x_p\in V(J)}\prod_{ij\in E(F)}\psi(x_i,x_j).$$

In terms of this definition expression (4) can be written as $2(m|J|)^p\sum_{F\subseteq G}\Psi(J;F)$, where the sum is over F with an even number of edges. Since $n=m|J|$ we deduce from (4) and (5) that

$$c(G;H) = 2^{1-e(G)}(1+o(1)) \sum_{F \subseteq G} \Psi(J;F),$$

the $o(1)$ term implying $m \to \infty$ but J fixed. We summarize the above discussion in the following lemma.

Lemma 9. *Let G and J be graphs, and for each spanning subgraph $F \subseteq G$, let $\Psi(J;F)$ be defined as above. Then*

$$c(G) \leq 2^{1-e(G)} \sum_{F \subseteq G} \Psi(J;F),$$

the sum being over all spanning subgraphs of G with an even number of edges.

Clearly, then, to show that a graph G is uncommon it is enough to find a graph J such that $\sum_{F \subseteq G} \Psi(J;F)$ is less than one. Note that $\Psi(J;F) = 1$ if F has no edges.

Let $k \geq 1$ be an integer, and let V be a $2k$-dimensional vector space over $GF(2)$ with a specified basis. We define a quadratic form q on V by

$$q(x) = q((\xi_1, \ldots, \xi_{2k})) = \xi_1^2 + \xi_2^2 + \xi_1\xi_2 + \xi_3\xi_4 + \ldots + \xi_{2k-1}\xi_{2k}.$$

(This is the quadratic form of minimal Witt index.) The equation $\langle x,y \rangle = q(x+y) + q(x) + q(y)$ defines an inner product on V; its formula in terms of coordinates is given by

$$\langle (\xi_1, \ldots, \xi_{2k}), (\eta_1, \ldots, \eta_{2k}) \rangle = \xi_1\eta_2 + \xi_2\eta_1 + \xi_3\eta_4 + \xi_4\eta_3 + \ldots + \xi_{2k-1}\eta_{2k} + \xi_{2k}\eta_{2k-1}.$$

The *orthogonal tower* graph T_k^- has vertex set V, and $xy \in E(T_k^-)$ if $q(x+y) = 1$. The order of this graph is therefore $|V| = 2^{2k}$. For any $z \in V$ the map $x \mapsto x+z$ defines an automorphism of the graph T_k^-, which is therefore vertex transitive. The degree of each vertex is equal to the number of $x \in V$ with $q(x) = 1$. This number can be calculated quite easily (by algebraic or combinatorial methods according to taste), and equals $2^{2k-1} + 2^{k-1}$. A more detailed description of the structure of T_k^- is given in [15].

We now define the complex valued function φ on V by $\varphi(x) = (-1)^{q(x)}$. Of course, here we are interpreting q as taking values in the complex numbers rather than $GF(2)$. Clearly, if we take J to be T_k^- in our earlier discussion then $\psi(x,y) = \varphi(x+y)$, so

$$\Psi(T_k^-;F) = \frac{1}{|V|^p} \sum_{x_1 \in V} \cdots \sum_{x_p \in V} \prod_{ij \in E(F)} \varphi(x_i + x_j).$$

By our earlier calculation of the degrees in T_k^-, we have

$$\frac{1}{|V|}\sum_x \varphi(x) = \frac{1}{|V|}\sum_x (-1)^{q(x)} = -(2^{2k-1}+2^{k-1}) + (2^{2k-1}-2^{k-1}) = -2^{-k}.$$

Furthermore, the inner product $\langle x,y\rangle$ is non-singular, so for $x\neq 0$ there are exactly $|V|/2$ values of y for which $\langle x,y\rangle=0$. Therefore

$$\frac{1}{|V|^2}\sum_{x,y\in V}(-1)^{\langle x,y\rangle} = \frac{1}{|V|^2}\sum_{y\in V}(-1)^{\langle 0,y\rangle} = \frac{1}{|V|} = (-2^{-k})^2.$$

It turns out that $\Psi(T_k^-;F)$ has a simple formulation in terms of the sum $|V|^{-1}\sum\limits_x \varphi(x)$. This sum can be regarded as the Fourier transform $\hat{\varphi}$ of φ evaluated at zero, and the calculation of $\Psi(T_k^-;F)$ can be phrased in these terms. However we shall adopt a basic approach and shall perform all calculations explicitly.

Lemma 10. *Let $r=r(F)$ be the rank of the $p\times p$ matrix $M=M(F)$ over $GF(2)$ defined by $m_{ij}=m_{ji}=1$ if $i\neq j$ and $ij\in E(F)$, $m_{ij}=m_{ji}=0$ if $i\neq j$ and $ij\notin E(F)$, and m_{ii} is chosen so that the row sums are zero. Then*

$$\Psi(T_k^-;F) = (-1)^r 2^{-kr}.$$

Proof. For $1\leq i<j\leq p$ we define E_{ij} to be the $p\times p$ matrix with four entries, namely the ii, ij, ji, and jj entries, equal to one and the other entries equal to zero. Observe that $M(F)=\sum\limits_{ij\in E(F)} E_{ij}$. By standard arguments from the theory of bilinear forms we know there exists a non-singular $p\times p$ matrix C over $GF(2)$ such that the matrix C^tMC is nearly diagonal. To be precise, for some integer s, $0\leq s\leq r/2$, we have $C^tMC=N=(n_{ij})$, where $n_{2i-1,2i}=n_{2i,2i-1}=1$ for $1\leq i\leq s$, $n_{ii}=1$ for $2s+1\leq i\leq r$ and all other entries of N are equal to zero. (In fact it can be further specified that $s=0$ or $s=r/2$.)

We wish to evaluate

$$\Psi(T_k^-;F) = \frac{1}{|V|^p}\sum_{x_1,\ldots,x_p\in V}\ \prod_{ij\in E(F)}\varphi(x_i+x_j)$$
$$= \frac{1}{|V|^p}\sum_{x_1,\ldots,x_p\in V}(-1)^{\sum\limits_{ij\in E(F)} q(x_i+x_j)} = \frac{1}{|V|^p}\sum_{\mathbf{x}\in V^p}(-1)^{Q(\mathbf{x})},$$

where $Q(\mathbf{x})$ is a quadratic form on V^p defined by

$$Q(\mathbf{x}) = Q((x_1,\ldots,x_p)) = \sum_{ij\in E(F)} q(x_i+x_j).$$

Let A be the $2k\times 2k$ matrix such that $q(x)=x^tAx$ for $x\in V$. Note that $q(x+y)=x^tAx+x^tAy+y^tAx+y^tAy$, and therefore that $\langle x,y\rangle = x^tAy+y^tAx$. Regarding V^p now as a $2kp$-dimensional vector space over $GF(2)$, the quadratic form Q has matrix $\sum_{ij\in F} E_{ij}\otimes A = M\otimes A$, where by the tensor product $M\otimes A$ we mean the $2kp\times 2kp$ matrix which is a $p\times p$ array of $2k\times 2k$ blocks, the ij block being $m_{ij}A$.

Since $C^tMC=N$ we have $(C\otimes I)^t(M\otimes A)(C\otimes I)=N\otimes A$ where I is the $2k\times 2k$ identity matrix. As $\mathbf{x}$ takes all values in V^p so does $(C\otimes I)\mathbf{x}$, and thus

$$\begin{aligned}\Psi(T_k^-;F) &= \frac{1}{|V|^p}\sum_{\mathbf{x}\in V^p}(-1)^{Q(\mathbf{x})} = \frac{1}{|V|^p}\sum_{\mathbf{x}\in V^p}(-1)^{\mathbf{x}^t(M\otimes A)\mathbf{x}}\\ &= \frac{1}{|V|^p}\sum_{\mathbf{x}\in V^p}(-1)^{\mathbf{x}^t(N\otimes A)\mathbf{x}}.\end{aligned}$$

Now if $\mathbf{x}=(x_1,\dots,x_p)$ then

$$\begin{aligned}\mathbf{x}^t(N\otimes A)\mathbf{x} &= \sum_{i=1}^{s}(x_{2i-1}^tAx_{2i}+x_{2i}^tAx_{2i-1}) + \sum_{i=2s+1}^{r} x_i^tAx_i\\ &= \sum_{i=1}^{s}\langle x_{2i-1},x_{2i}\rangle + \sum_{i=2s+1}^{r} q(x_i).\end{aligned}$$

Therefore

$$\begin{aligned}\Psi(T_k^-;F) &= \frac{1}{|V|^p}\sum_{x_1\in V}\cdots\sum_{x_p\in V}(-1)^{\sum_{i=1}^{s}\langle x_{2i-1},x_{2i}\rangle+\sum_{i=2s+1}^{r}q(x_i)}\\ &= \left(\frac{1}{|V|^2}\sum_{x,y\in V}(-1)^{\langle x,y\rangle}\right)^s\left(\frac{1}{|V|}\sum_{x\in V}(-1)^{q(x)}\right)^{r-2s}\left(\frac{1}{|V|}\sum_{x\in V}1\right)^{p-r}\\ &= \left(-2^{-k}\right)^r\end{aligned}$$

by the calculations immediately preceding the lemma. ∎

It is very easy to calculate which graphs F have matrices $M(F)$ of low rank.

Lemma 11. *The matrix $M=M(F)$ defined in the previous lemma has rank* 0 *if and only if F is empty, and has rank* 1 *if and only if F is a complete graph of even order together with isolated vertices.*

Proof. Isolated vertices correspond to zero rows (and columns) of $M(F)$ and so can be ignored; let us assume then that $M(F)$ has no zero rows or columns. If rank $M=1$ then the rows of M are identical. Since every column has a non-zero entry it must therefore consist entirely of ones; thus all the entries of M are ones. The row sums of M are even so M has an even number of columns. Hence the (non-isolated) vertices of F span a complete graph of even order. ∎

We can now prove our main theorem.

Theorem 12. *Every graph containing K_4 is uncommon.*

Proof. Let G be a graph. By Lemmas 9 and 10,

$$c(G) \leq 2^{1-e(G)} \sum_{F \subseteq G} (-1)^{r(F)} 2^{-kr(F)} = 2^{1-e(G)} \sum_{j \geq 0} g_j (-2^k)^j,$$

where g_j is the number of subgraphs $F \subseteq G$ with even size satisfying $r(F)=j$. By Lemma 11, $g_0=1$ and g_1 counts the number of subgraphs of G which are complete with both even order and even size; that is, g_1 is the number of complete subgraphs whose orders are divisible by four. Therefore, $g_1>0$ if and only if $G \supset K_4$. But the inequality $c(G) \leq 2^{1-e(G)} \sum_{j\geq 0} g_j(-2^k)^j$ holds for all $k \geq 1$, and clearly, if $g_1>0$ and k is large enough, the sum is less than one. This proves the theorem. ∎

Remark. For the case $G=K_p$ it was shown in [15] that $c(G) < 0.936 \times 2^{1-e(G)}$ if $p \geq 6$. This bound can be improved by doing the above calculations more precisely for complete graphs G. Specifically, it is possible, by extending Lemma 11, to calculate g_1 and g_2 exactly (it is necessary to consider the congruence class of r (mod 8)). Upper bounds on g_j for $j \geq 3$ can be estimated quite easily, and then setting $k=p$ the bound $c(G) < 0.835 \times 2^{1-e(G)}$ can be obtained. No better lower bound than the trivial one given by Ramsey numbers is known in general, although Giraud [6] showed that $c(K_4) > 0.695 \times 2^{1-e(G)}$.

5. More than two colours

We conclude with some observations about multiplicities of monochromatic subgraphs when $s \geq 3$ colours are used to colour the edges of K_n. The equivalent definition of "common" is that a graph G is s-common if the proportion of subgraphs of K_n isomorphic to G which are monochromatic is (asymptotically) the proportion obtained when the colouring is random, the colour of each edge being chosen uniformly from a set of s colours; the proportion obtained randomly is of course $s^{1-e(G)}$.

First, let us note that, if Conjecture 1 is true, then any bipartite graph is s-common for every $s \geq 2$. This follows from a simple convexity argument as for the case when $s=2$. We know of no non-bipartite graph which is s-common for any $s \geq 3$. It would be interesting to know if there are any; in view of the next result the case $s=3$ is the critical one.

Theorem 13. *If a graph is s-uncommon then it is $(s+1)$-uncommon.*

Proof. Let G be s-uncommon, and take a sequence of colourings of K_n which demonstrate this fact. That is, the proportion of monochromatic labelled subgraphs isomorphic to G in these colourings tends to $cs^{1-e(G)}$ for some constant $c<1$. In each colouring, select at random $1/(s+1)$ of the edges and colour them with a new colour. The number of monochromatic copies of G in this new colouring is therefore asymptotically equal to

$$cs^{1-e(G)}n^{|G|}\left(\frac{s}{s+1}\right)^{e(G)}+n^{|G|}\left(\frac{1}{s+1}\right)^{e(G)}=\frac{n^{|G|}}{(s+1)^{e(G)-1}}\left(c\frac{s}{s+1}+\frac{1}{s+1}\right),$$

showing that G is $(s+1)$-uncommon. ∎

We now show that every non-bipartite graph is s-uncommon if s is large enough. The colourings we use to demonstrate this are all very straightforward, involving partitioning the vertices into blocks such that all edges between two blocks have the same colour.

Theorem 14. *Let G be a non-bipartite graph. Then G is s-uncommon provided s is large enough (specifically, provided $2^{2s-2}\geq s^{|G|}$).*

Proof. Let $s=k+1$, and let $\{v_1,\dots,v_{2^k}\}$ be the vertex set of a complete graph of order 2^k. It is easy to verify that the edges of this graph can be k-coloured so that each colour induces a bipartite subgraph. We now colour K_n by partitioning the vertex set into 2^k sets $V_1,\dots,V_{2^k}$, each of size $\lfloor n2^{-k}\rfloor$ or $\lceil n2^{-k}\rceil$, and colouring all edges between V_i and V_j the same colour as the edge v_iv_j, $1\leq i<j\leq 2^k$. The edges within each V_i are coloured with the $(k+1)$st colour.

Clearly the only monochromatic copies of G in this colouring are those with the $(k+1)$st colour, of which there are $2^k(n/2^k)^{|G|}(1+o(1))$. This quantity is less than $s^{1-e(G)}n^{|G|}(1+o(1))$ provided $2^{k(1-|G|)}<s^{1-e(G)}$. Since $e(G)\leq\binom{|G|}{2}$ this inequality will hold provided $s^{|G|}\leq 2^{2k}=2^{2s-2}$, as claimed. ∎

Of course, for a particular G, more care can be taken with the calculations in the proof of Theorem 14, and indeed the colouring can be adapted slightly. Using block colourings in this way it can be shown that all odd cycles are 3-uncommon (see Jagger [8] for details). We do not know whether the 3-fan is 3-uncommon.

6. Summary

We have shown that, for any graph G containing K_4, there is a two-colouring of K_n containing fewer monochromatic copies of G than found in a random colouring. It is conjectured that for bipartite graphs there are no colourings which do better than random colourings. The main question we leave unanswered is whether the condition that G contains K_4 can be strengthened to the condition that the chromatic number of G is at least four. In particular, what happens when G is the 5-wheel?

References

[1] S. A. Burr and V. Rosta: On the Ramsey multiplicities of graph problems and recent results, *J. Graph Theory*, **4**, (1980) 347–361.

[2] P. Erdős: On the number of complete subgraphs contained in certain graphs, *Publ. Math. Inst. Hung. Acad. Sci.,* VII, Ser. A3 (1962) 459–464.

[3] P. Erdős and J. W. Moon: On subgraphs of the complete bipartite graph, *Canad. Math. Bull.* **7** (1964) 35–39.

[4] P. Erdős and M. Simonovits: Supersaturated graphs and hypergraphs, *Combinatorica* **3** (2) (1983) 181–192.

[5] F. Franek and V. Rödl: 2-Colorings of complete graphs with a small number of monochromatic K_4 subgraphs, *Discrete Mathematics*, **114**, (1993) 199–203.

[6] G. Giraud: Sur le probléme de Goodman pour les quadrangles et la majoration des nombre de Ramsey, *J. Combin. Theory, Series B*, **27**, (1979) 237–253.

[7] A. W. Goodman: On sets of acquaintances and strangers at any party, *Amer. Math. Monthly*, **66** (1959), 778–783.

[8] C. N. Jagger: Ph.D. thesis, (in preparation).

[9] G. Lorden: Blue-empty chromatic graphs, *Amer. Math. Monthly*, **69** (1962), 114–120.

[10] A. F. Sidorenko: Cycles in graphs and functional inequalities, *Mathematical Notes*, **46** (1989), 877–882.

[11] A. F. Sidorenko: Extremal problems for finite sets, *Bolyai Society Mathematical Studies*, **3** (1994), 425–457.

[12] A. F. Sidorenko: A correlation inequality for bipartite graphs, *Graphs and Combinatorics*, **9** (1993), 201–204.

[13] A. F. Sidorenko: Inequalities for functionals generated by bipartite graphs, *Discrete Math. Appl.*, **2** (5) (1992), 489–504.

[14] A. F. Sidorenko: Extremal problems in graph theory and inequalities in functional analysis, *Proc. Soviet Seminar on Discrete Math. Appl.* Moscow Univ Press, 1986, 99–105 (in Russian).

[15] A. G. THOMASON: A disproof of a conjecture of Erdős in Ramsey Theory, *J. London Math. Soc.*, **39** (2) (1989), 246–255.

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